Lecture 22

In this lecture we continue our study of predicate logic, looking at negations and at how quantifiers can be combined.

$\forall$ — for <u>a</u>ll

$\exists$ — there <u>e</u>xists

<u>E.g.</u> "Every student has brains."

Let D = the class of all students.

This is a set — the domain of interpretation.

So "every student has brains" becomes:

$$\forall x \in D, \ x \text{ has brains}$$

Let $P(x)$ be the predicate:

"student x has brains".

Then "every student has brains" becomes:

$$\forall x \in D, \ P(x) \text{ is true}$$

or, more simply,

$$\forall x \in D, \ P(x).$$

When we don't need to specify the domain, this becomes:

$$\forall x \ P(x)$$

How do we say, "There exists a student with brains"?

It is this:

$$\exists x \ P(x)$$

How do we say, "Some students have brains"?

It's the same: $\exists x \ P(x)$

In logic, "some" means "at least one".

This is how we apply one of the <u>quantifiers</u> $\forall$ and $\exists$ to a predicate.

Recall:

$\forall$ is the <u>universal</u> quantifier

$\exists$ is the <u>existential</u> quantifier

A statement involving predicates whose variables are all properly quantified becomes a proposition (provided that the domain is known).

<u>Eg.</u> $D = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$P(x)$ means "x is positive"

Which are true? $\forall x\, P(x)$, $\exists x\, P(x)$

<u>Sol$^{\underline{n}}$</u>

Consider $\forall x\, P(x)$.

This means "for all x, $P(x)$"

which means "for all x, x is positive".

This is false. A counterexample is $x = 0$ (or $x = -1$, etc).

(A <u>counterexample</u> is an example which proves that a universally quantified statement is false.)

Consider $\exists x\, P(x)$.

This means "there exists x <u>such that</u> $P(x)$ is true"

which means "there exists x such that x is positive".

This is true. For example, $x = 1$.

Note

It's more likely (loosely speaking) that

$$\exists x\, P(x) \text{ is true}$$

than that $\forall x\, P(x)$ is true.

In fact, provided that the domain D is nonempty,

$$\forall x\, P(x) \Rightarrow \exists x\, P(x).$$

Negations

What happens if we negate an expression involving predicates and quantifiers?

The Generalised de Morgan Laws

$$\neg\left[\forall x\ P(x)\right] \equiv \exists x\ \neg P(x)$$

$$\neg\left[\exists x\ P(x)\right] \equiv \forall x\ \neg P(x)$$

__E.g.__ "It's not true that all camels have humps" is the same as "there exists a camel that doesn't have a hump".

E.g. "It's not true that some dogs fly" is the same as "there aren't any dogs who fly" or equivalenty "all dogs don't fly".

Combining quantifiers

A predicate can have numerous variables, each of which may be quantified.

E.g.

$$\forall x \, \exists y \, \forall z \; P(x, y, z)$$

What's the negation?

$$\neg \left[\forall x \, \exists y \, \forall z \; P(x,y,z) \right] \equiv \exists x \, \forall y \, \exists z \; \neg P(x,y,z)$$

It can be difficult to interpret expressions involving 3 or more quantifiers.

Having 2 is a lot easier :

$$\forall x \, \forall y \; P(x,y)$$
$$\forall x \, \exists y \; P(x,y)$$
$$\exists x \, \forall y \; P(x,y)$$
$$\exists x \, \exists y \; P(x,y)$$
$$\forall y \, \exists x \; P(x,y)$$
$$\exists y \, \forall x \; P(x,y)$$

What do these mean? Which are true in any given situation?

This depends on how $P(x,y)$ is defined, and on what set is chosen as the domain of interpretation D.

<u>E.g.</u>

Suppose $P(x,y)$ means $x \geqslant y$.

Let D be the set $\mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\}$.

Which of the six predicate formulae given above are true?

<u>Solution</u>

Do as exercise.